GATE PATHSHALA

Fluid Mechanics and Aerodynamics (Assignment-02: Solutions)

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April 6, 2025

Total Acceleration, Convective Acceleration, and Angular velocity

Problem 1

For the velocity field:

$$\mathbf{V} = 2xy\hat{i} + 4tz^2\hat{j} - yz\hat{k}$$

find the acceleration, the angular velocity about the z-axis, and the vorticity vector at the point $(2, -1, 1)$ at $t = 2$.

Solution

Compute Acceleration

The total acceleration is given by:

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}$$

where:
- $\frac{\partial \mathbf{V}}{\partial t}$ is the **local acceleration**. - $(\mathbf{V} \cdot \nabla)\mathbf{V}$ is the **convective acceleration**.

Local Acceleration:

$$\frac{\partial \mathbf{V}}{\partial t} = 0\hat{i} + 4z^2\hat{j} + 0\hat{k}$$

Convective Acceleration:

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = (4xy^2 + 8txz^2)\hat{i} + (-8tyz^2)\hat{j} + (-4tz^3 + y^2z)\hat{k}$$

At $(x, y, z, t) = (2, -1, 1, 2)$:

$$\mathbf{a} = 40\hat{i} + 20\hat{j} - 7\hat{k}$$

Compute Vorticity and Angular Velocity

The vorticity vector is:

$$\boldsymbol{\zeta} = \nabla \times \mathbf{V}$$

$$\boldsymbol{\zeta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & 4tz^2 & -yz \end{vmatrix}$$

Expanding the determinant:

$$\boldsymbol{\zeta} = \hat{i}\left(\frac{\partial(-yz)}{\partial y} - \frac{\partial(4tz^2)}{\partial z}\right) + \hat{j}\left(\frac{\partial(2xy)}{\partial z} - \frac{\partial(-yz)}{\partial x}\right) + \hat{k}\left(\frac{\partial(4tz^2)}{\partial x} - \frac{\partial(2xy)}{\partial y}\right)$$

Computing derivatives:

$$\frac{\partial(-yz)}{\partial y} = -z, \quad \frac{\partial(4tz^2)}{\partial z} = 8tz$$

$$\frac{\partial(2xy)}{\partial z} = 0, \quad \frac{\partial(-yz)}{\partial x} = 0$$

$$\frac{\partial(4tz^2)}{\partial x} = 0, \quad \frac{\partial(2xy)}{\partial y} = 2x$$

Thus,

$$\boldsymbol{\zeta} = (-z - 8tz)\hat{i} + (0 - 0)\hat{j} + (0 - 2x)\hat{k}$$

$$\boldsymbol{\zeta} = (-z - 8tz)\hat{i} - 2x\hat{k}$$

At $(x, y, z, t) = (2, -1, 1, 2)$:

$$\boldsymbol{\zeta} = (-1 - 16)\hat{i} - 4\hat{k}$$

$$\boldsymbol{\zeta} = -17\hat{i} - 4\hat{k}$$

The angular velocity vector is given by:

$$\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\zeta}$$

$$\boldsymbol{\Omega} = \frac{1}{2}(-17\hat{i} - 4\hat{k})$$

$$\boldsymbol{\Omega} = -8.5\hat{i} - 2\hat{k}$$

Final Answers

Acceleration:

$$\mathbf{a} = 40\hat{i} + 20\hat{j} - 7\hat{k}$$

Vorticity:

$$\zeta = -17\hat{i} - 4\hat{k}$$

Angular velocity:

$$\boldsymbol{\Omega} = -8.5\hat{i} - 2\hat{k}$$

Problem 2

What is the equation of the streamline that passes through the point $(2, -1)$ when $t = 2$ s if the velocity field is given by:

(a) $\mathbf{V} = 2xy\mathbf{i} + y^2t\mathbf{j}$ m/s

(b) $\mathbf{V} = 2y^2\mathbf{i} + xyt\mathbf{j}$ m/s

Solution

Understanding the Streamline Equation

The equation of a streamline is given by:

$$\frac{dx}{u} = \frac{dy}{v}$$

where u and v are the velocity components in the x- and y-directions, respectively.

Consider Each Case Separately

Case (a): Velocity Field $\mathbf{V} = (2xy)\mathbf{i} + (y^2t)\mathbf{j}$

Here, the velocity components are:

$$u = 2xy,$$
$$v = y^2t.$$

At $t = 2$, the velocity components become:

$$u = 2xy,$$
$$v = 2y^2.$$

The streamline equation:

$$\frac{dx}{2xy} = \frac{dy}{2y^2}$$

Canceling common terms:

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

Rearrange:

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrate both sides:

$$\ln|x| = \ln|y| + C$$

or

$$x = Cy.$$

Using the initial condition $(x, y) = (2, -1)$:

$$2 = C(-1)$$

$$C = -2.$$

Thus, the streamline equation is:

$$x = -2y.$$

Case (b): Velocity Field $\mathbf{V} = (2y^2)\mathbf{i} + (xyt)\mathbf{j}$

Here, the velocity components are:

$$u = 2y^2,$$
$$v = xyt.$$

At $t = 2$, the velocity components become:

$$u = 2y^2,$$
$$v = 2xy.$$

The streamline equation:

$$\frac{dx}{2y^2} = \frac{dy}{2xy}$$

Canceling 2:

$$\frac{dx}{y^2} = \frac{dy}{xy}$$

Rearrange:

$$x\,dx = y\,dy.$$

Integrate both sides:

$$\frac{x^2}{2} = \frac{y^2}{2} + C.$$

or

$$x^2 - y^2 = C.$$

Using the initial condition $(x, y) = (2, -1)$:

$$2^2 - (-1)^2 = C$$

$$4 - 1 = 3.$$

Thus, the streamline equation is:

$$x^2 - y^2 = 3.$$

Final Answer

- For **Case (a)**: $x = -2y$
- For **Case (b)**: $x^2 - y^2 = 3$

Problem 3

Write all the non-zero terms of $D\rho/Dt$ for a stratified flow in which:

(a) $\rho = \rho(z)$ and $\mathbf{V} = z(2 - z)\mathbf{i}$

(b) $\rho = \rho(z)$ and $\mathbf{V} = f(x, z)\mathbf{i} + g(x, z)\mathbf{j}$

Solution

Understanding the Material Derivative

The material derivative of density, $D\rho/Dt$, is given by:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho$$

Since the problem specifies a stratified flow where $\rho = \rho(z)$, the density does not explicitly depend on time or x, y. This simplifies the material derivative to:

$$\frac{D\rho}{Dt} = \mathbf{V} \cdot \nabla \rho$$

where the gradient of density is:

$$\nabla \rho = \frac{d\rho}{dz}\mathbf{k}$$

Thus, the material derivative simplifies further to:

$$\frac{D\rho}{Dt} = w\frac{d\rho}{dz}$$

where w is the velocity component in the z-direction.

Case (a)

Given:

$$\mathbf{V} = z(2 - z)\mathbf{i}$$

This velocity field has only an x-component, meaning:

$$w = 0.$$

Since the material derivative depends on the z-component of velocity and there is none, we conclude:

$$\frac{D\rho}{Dt} = 0.$$

Case (b)

Given:

$$\mathbf{V} = f(x, z)\mathbf{i} + g(x, z)\mathbf{j}$$

Since there is no k-component ($w = 0$), we again conclude:

$$\frac{D\rho}{Dt} = 0.$$

Final Answer

For both cases (a) and (b), the material derivative of density is zero:

$$\frac{D\rho}{Dt} = 0.$$

Problem 4

Decide if each of the following can be modeled as an incompressible flow or a compressible flow:

(a) The take-off and landing of commercial airplanes

(b) The airflow around an automobile

(c) The flow of air in a hurricane

(d) The airflow around a baseball thrown at 100 mi/h

Solution

To determine whether each case can be modeled as incompressible or compressible flow, we use the Mach number (M):

$$M = \frac{V}{c}$$

where:

- V is the flow velocity,
- c is the speed of sound in air (≈ 343 m/s at sea level).

A flow is typically considered **compressible** if $M \geq 0.3$, since significant density changes occur beyond this threshold. Otherwise, it is considered **incompressible**.

(a) Take-off and landing of commercial airplanes

- Typical speeds: **150–250 mi/h** (67–112 m/s)
- Mach number: $M = \frac{112}{343} \approx 0.33$ (for upper limit)
- Since $M \approx 0.33$, compressibility effects might start to appear, but for most engineering purposes, this can be **modeled as incompressible flow**.

(b) Airflow around an automobile

- Typical speeds: **30–80 mi/h** (13–36 m/s)
- Mach number: $M = \frac{36}{343} \approx 0.1$
- Since $M \ll 0.3$, the flow can be **modeled as incompressible**.

(c) Flow of air in a hurricane

- Typical wind speeds: **75–200 mi/h** (34–89 m/s)
- Mach number: $M = \frac{89}{343} \approx 0.26$
- Since $M < 0.3$, compressibility effects are negligible. This can be **modeled as incompressible flow**.

(d) Airflow around a baseball thrown at 100 mi/h

- Speed: **100 mi/h** (45 m/s)
- Mach number: $M = \frac{45}{343} \approx 0.13$
- Since $M \ll 0.3$, this can be **modeled as incompressible flow**.

Final Answer

Case	Flow Type
(a) Take-off and landing of commercial airplanes	Incompressible
(b) Airflow around an automobile	Incompressible
(c) Flow of air in a hurricane	Incompressible
(d) Airflow around a baseball at 100 mi/h	Incompressible

Since all cases have $M < 0.3$, they can all be **modeled as incompressible flows**.

Problem 5

Select the word: **uniform, one-dimensional, two-dimensional, or three-dimensional**, which best describes each of the following flows:

(a) Developed flow in a pipe

(b) Flow of water over a long weir

(c) Flow in a long, straight canal

(d) The flow of exhaust gases exiting a rocket

(e) Flow of blood in an artery

(f) Flow of air around a bullet

(g) Flow of blood in a vein

(h) Flow of air in a tornado

Solution

Based on the characteristics of the flow, we classify each case as follows:

Flow Type	Classification
(a) Developed flow in a pipe	One-Dimensional (1D)
(b) Flow of water over a long weir	Two-Dimensional (2D)
(c) Flow in a long, straight canal	Two-Dimensional (2D)
(d) The flow of exhaust gases exiting a rocket	One-Dimensional (1D)
(e) Flow of blood in an artery	One-Dimensional (1D)
(f) Flow of air around a bullet	Three-Dimensional (3D)
(g) Flow of blood in a vein	One-Dimensional (1D)
(h) Flow of air in a tornado	Three-Dimensional (3D)

Explanation

- **(a) Developed Flow in a Pipe → 1D**
  Velocity varies radially but remains uniform along the pipe axis in fully developed flow.

- **(b) Flow of Water Over a Long Weir → 2D**
  The velocity varies in both vertical and horizontal directions.

- **(c) Flow in a Long, Straight Canal → 2D**
  The flow is primarily in one direction, but velocity changes with depth.

- **(d) Flow of Exhaust Gases Exiting a Rocket → 1D**
  Assuming a steady and streamlined exit, velocity varies only in one direction.

- **(e) Flow of Blood in an Artery → 1D**
  Similar to pipe flow, blood flow is primarily along the artery's axis.

- **(f) Flow of Air Around a Bullet → 3D**
  The flow field around a moving bullet varies in all three spatial directions.

- **(g) Flow of Blood in a Vein → 1D**
  Similar to an artery, blood flow is mostly along the length of the vein.

- **(h) Flow of Air in a Tornado → 3D**
  The swirling motion creates velocity variations in all three spatial directions.

Problem 6

To determine the rate of change of temperature of a fluid particle, we use the material derivative:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T$$

Given:

- Velocity field:
$$\mathbf{V} = 2y\mathbf{i} + x\mathbf{j} + t\mathbf{k}$$

- Temperature field:
$$T(x, y, z) = 20xy$$

- Point: $(x, y, z) = (2, 1, -2)$ at $t = 2$

Solution

Compute the Temperature Gradient ∇T

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

$$\frac{\partial T}{\partial x} = 20y, \quad \frac{\partial T}{\partial y} = 20x, \quad \frac{\partial T}{\partial z} = 0$$

Thus,

$$\nabla T = (20y, 20x, 0)$$

At the point $(x, y, z) = (2, 1, -2)$:

$$\nabla T = (20(1), 20(2), 0) = (20, 40, 0)$$

Compute the Convective Term $\mathbf{V} \cdot \nabla T$

$$\mathbf{V} \cdot \nabla T = (2y, x, t) \cdot (20, 40, 0)$$

$$= (2(1) \cdot 20) + (2 \cdot 40) + (-2 \cdot 0)$$

$$= 40 + 80 + 0 = 120$$

Compute the Total Rate of Change

Since $T(x, y, z)$ is not explicitly dependent on t, we have:

$$\frac{\partial T}{\partial t} = 0$$

Thus,

$$\frac{DT}{Dt} = 0 + 120 = 120\,^{\circ}C/s$$

Problem 7

Determine the velocity V in the pipe if the fluid in the pipe of Figure(p7) is:

(a) Atmospheric air and $h = 40$ cm of water

(b) Water and $h = 20$ cm of mercury

(c) Kerosene and $h = 30$ cm of mercury

(d) Gasoline and $h = 80$ cm of water

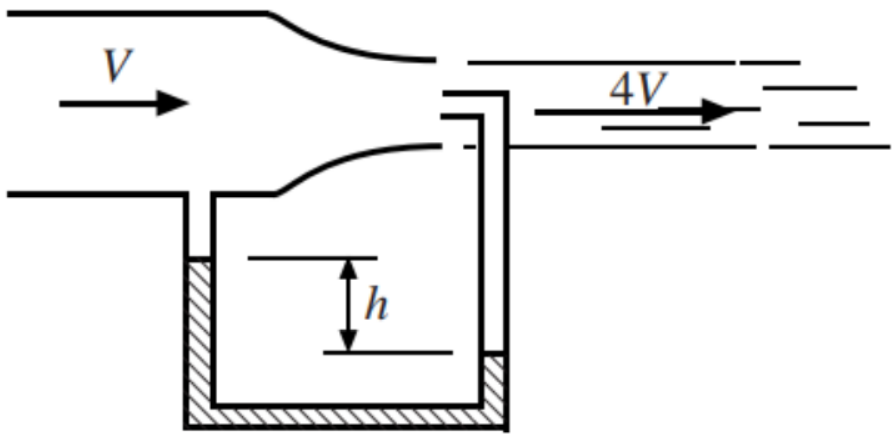

Figure 1:

Solution

Using Bernoulli's equation between two points inside the pipe:

$$P_1 + \frac{1}{2}\rho V^2 = P_2 + \frac{1}{2}\rho(4V)^2$$

Rearrange:

$$P_1 - P_2 = \frac{1}{2}\rho(16V^2 - V^2) = \frac{15}{2}\rho V^2$$

The pressure difference is also given by the manometer equation:

$$P_1 - P_2 = \rho_f g h$$

Equating both expressions:

$$\rho_f g h = \frac{15}{2}\rho V^2$$

Solving for V:

$$V = \sqrt{\frac{2\rho_f g h}{15\rho}}$$

where:

- $\rho = 1.225 \text{ kg/m}^3$ (air density)
- $g = 9.81 \text{ m/s}^2$
- ρ_f depends on the fluid used
- h is the height of the liquid column

(a) Atmospheric air, $h = 40$ cm of water

$$V = \sqrt{\frac{2(1000)(9.81)(0.40)}{15(1.225)}} = \sqrt{427.2} = \mathbf{20.67} \text{ m/s}$$

(b) Water, $h = 20$ cm of mercury

$$V = \sqrt{\frac{2(13560)(9.81)(0.20)}{15(1000)}} = \sqrt{3.54} = \mathbf{1.881} \text{ m/s}$$

(c) Kerosene, $h = 30$ cm of mercury

$$V = \sqrt{\frac{2(13560)(9.81)(0.30)}{15(800)}} = \sqrt{6.65} = \mathbf{2.57} \text{ m/s}$$

(d) Gasoline, $h = 80$ cm of water

$$V = \sqrt{\frac{2(1000)(9.81)(0.80)}{15(740)}} = \sqrt{1.41} = \mathbf{1.18} \text{ m/s}$$

Problem 8

A velocity field is given in cylindrical coordinates as:

$$v_r = \left(4 - \frac{1}{r^2}\right)\sin\theta \quad \text{m/s},$$
$$v_\theta = -\left(4 + \frac{1}{r^2}\right)\cos\theta \quad \text{m/s},$$
$$v_z = 0.$$

Find:

1. The acceleration at the point $(0.6 \text{ m}, 90^\circ)$.
2. The vorticity at the point $(0.6 \text{ m}, 90^\circ)$.

Solution

Compute Velocity at Given Point

Substituting $r = 0.6$ and $\theta = 90°$:

$$\sin 90° = 1, \quad \cos 90° = 0,$$
$$v_r = \left(4 - \frac{1}{0.6^2}\right) \times 1 = 4 - \frac{1}{0.36} = 4 - 2.78 = 1.22 \text{ m/s},$$
$$v_\theta = -\left(4 + \frac{1}{0.6^2}\right) \times 0 = 0.$$

Thus, the velocity vector at this point is:

$$\mathbf{V} = (1.22\hat{e}_r + 0\hat{e}_\theta + 0\hat{e}_z) \text{ m/s}.$$

Compute Acceleration

The material derivatives for acceleration in cylindrical coordinates are:

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r},$$
$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r}.$$

Partial Derivatives of v_r

$$\frac{\partial v_r}{\partial r} = \left(\frac{2}{r^3}\right)\sin\theta, \quad \frac{\partial v_r}{\partial \theta} = \left(4 - \frac{1}{r^2}\right)\cos\theta.$$

Substituting $r = 0.6$ and $\theta = 90°$:

$$\frac{\partial v_r}{\partial r} = \frac{2}{(0.6)^3} \times 1 = \frac{2}{0.216} = 9.26,$$
$$\frac{\partial v_r}{\partial \theta} = \left(4 - \frac{1}{(0.6)^2}\right) \times 0 = 0.$$

Partial Derivatives of v_θ

$$\frac{\partial v_\theta}{\partial r} = -\left(\frac{2}{r^3}\right)\cos\theta, \quad \frac{\partial v_\theta}{\partial \theta} = \left(4 + \frac{1}{r^2}\right)\sin\theta.$$

Substituting values:

$$\frac{\partial v_\theta}{\partial r} = -\frac{2}{(0.6)^3} \times 0 = 0,$$
$$\frac{\partial v_\theta}{\partial \theta} = \left(4 + \frac{1}{(0.6)^2}\right) \times 1 = 4 + 2.78 = 6.78.$$

Using $v_\theta = 0$, the acceleration simplifies:

$$a_r = v_r \frac{\partial v_r}{\partial r} = (1.22)(9.26) = 11.3 \text{ m/s}^2,$$
$$a_\theta = v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} = (1.22)(0) + 0 = 0.$$

Thus, the acceleration is:

$$\mathbf{a} = 11.3\hat{e}_r \text{ m/s}^2.$$

Compute Vorticity

The vorticity is given by:

$$\boldsymbol{\omega} = \nabla \times \mathbf{V}.$$

The only nonzero component is:

$$\omega_z = \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right)$$

Substituting values:

$$\omega_z = \frac{1}{0.6}(0) - (0) = 0.$$

Thus, the vorticity is:

$$\boldsymbol{\omega} = 0.$$

Final Answers:

- **Acceleration:** $11.3\hat{e}_r$ m/s^2
- **Vorticity:** 0 rad/s