GATE PATHSHALA
Integrals

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Properties of Indefinite Integrals

Let $f(x)$, $g(x)$ be integrable functions and $c \in \mathbb{R}$:

1. Linearity

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$
$$\int c \cdot f(x)\, dx = c \cdot \int f(x)\, dx$$

2. Reversal of Derivative

$$\frac{d}{dx}\left(\int f(x)\, dx\right) = f(x)$$

3. General Antiderivative

$$\int f(x)\, dx = F(x) + C, \quad \text{where } F'(x) = f(x)$$

4. Substitution Rule

If $u = g(x)$, then:

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du$$

List of Indefinite Integrals

A. Basic Power and Polynomial Forms

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$
$$\int \frac{1}{x}\, dx = \ln|x| + C$$
$$\int \sqrt{x}\, dx = \frac{2}{3}x^{3/2} + C$$

B. Exponential and Logarithmic Functions

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$$

$$\int \ln x \, dx = x \ln x - x + C$$

C. Trigonometric Functions

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

D. Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$$

$$\int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x + C$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\int \frac{-1}{1+x^2} \, dx = \cot^{-1} x + C$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$$

$$\int \frac{-1}{|x|\sqrt{x^2-1}} \, dx = \csc^{-1} x + C$$

E. Hyperbolic Functions

$$\int \sinh x \, dx = \cosh x + C$$
$$\int \cosh x \, dx = \sinh x + C$$
$$\int \tanh x \, dx = \ln|\cosh x| + C$$
$$\int \text{sech}^2 x \, dx = \tanh x + C$$
$$\int \text{csch}^2 x \, dx = -\coth x + C$$

F. Other Common Forms

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

Properties of Definite Integrals

Let $f(x)$, $g(x)$ be integrable functions on $[a, b]$, and $c \in \mathbb{R}$:

1. Linearity

$$\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$$
$$\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

2. Interval Additivity

$$\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx$$

3. Reversal of Limits

$$\int_b^a f(x) \, dx = -\int_a^b f(x) \, dx$$

4. Zero Interval

$$\int_a^a f(x) \, dx = 0$$

5. Comparison Property

If $f(x) \leq g(x)$ on $[a, b]$, then:

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

6. Absolute Value Inequality

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

7. Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then $\exists c \in [a, b]$ such that:

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

8. Fundamental Theorem of Calculus

If $F'(x) = f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Common Definite Integrals

1. Power and Polynomial Functions

$$\int_0^1 x^n\,dx = \frac{1}{n+1}, \quad n > -1$$

$$\int_{-a}^a x^n\,dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2a^{n+1}}{n+1} & \text{if } n \text{ is even} \end{cases}$$

2. Exponential Functions

$$\int_0^1 e^x\,dx = e - 1$$

$$\int_0^\infty e^{-x}\,dx = 1$$

$$\int_0^\infty xe^{-x}\,dx = 1$$

$$\int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1) \quad \text{(Gamma function)}$$

3. Logarithmic Functions

$$\int_0^1 \ln x \, dx = -1$$

$$\int_1^a \ln x \, dx = a \ln a - a + 1$$

4. Trigonometric Functions

$$\int_0^\pi \sin x \, dx = 2, \qquad \int_0^\pi \cos x \, dx = 0$$

$$\int_0^{2\pi} \sin^2 x \, dx = \pi, \qquad \int_0^{2\pi} \cos^2 x \, dx = \pi$$

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{(n-1)!!}{n!!} & \text{if } n \text{ is odd} \end{cases}$$

5. Inverse Trigonometric Functions

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2}$$

$$\int_0^1 \tan^{-1} x \, dx = \frac{\pi \ln 2}{4}$$

6. Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

7. Special Definite Integrals

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2} \quad \text{(Dirichlet integral)}$$

$$\int_0^\infty \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2}$$

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}$$

$$\int_0^1 \frac{x^{p-1}}{1+x} \, dx = \frac{\pi}{\sin(\pi p)} \quad \text{for } 0 < p < 1$$

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