Orthogonal and Orthonormal vectors

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Difference Between Orthogonal and Orthonormal Vectors

1. Orthogonal Vectors

Two vectors are **orthogonal** if their dot product is **zero**. Mathematically, for vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} \cdot \mathbf{b} = 0 \tag{1}$$

Example:

Let $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (-2, 1)$:

$$\mathbf{a} \cdot \mathbf{b} = (1)(-2) + (2)(1) = -2 + 2 = 0. \tag{2}$$

Since the dot product is zero, $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

2. Orthonormal Vectors

Two vectors are **orthonormal** if they are **both orthogonal and of unit length**. This means:

$$\mathbf{a} \cdot \mathbf{b} = 0, \quad \text{and} \quad \|\mathbf{a}\| = 1, \quad \|\mathbf{b}\| = 1. \tag{3}$$

Example: Let $\mathbf{a} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\mathbf{b} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$:

$$\mathbf{a} \cdot \mathbf{b} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{2} = 0. \tag{4}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and their magnitudes are:

$$\|\mathbf{a}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1, \tag{5}$$

$$\|\mathbf{b}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1. \tag{6}$$

Since both vectors have unit length, they are **orthonormal**.