GATE PATHSHALA

Laplace Transform: Basics and Solved Examples

Author Name: Mohd Abrar Nizami

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Definition

The **Laplace transform** of a function $f(t)$, defined for $t \geq 0$, is given by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

This transformation converts a time-domain function $f(t)$ into a complex frequency-domain function $F(s)$, which is often easier to manipulate for solving differential equations.

Basic Properties

1. **Linearity:**
$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$
2. **Time Shifting:**
$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$
3. **Frequency Shifting:**
$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$
4. **Differentiation in Time Domain:**
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
5. **Integration in Time Domain:**
$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}$$

Solved Examples

1. $f(t) = 1$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^\infty = \frac{1}{s}$$

2. $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,dt = \frac{1}{s^2}$$

3. $f(t) = t^n$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n \in \mathbb{N}_0$$

   Example:

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$$

4. $f(t) = e^{at}$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad \mathrm{Re}(s) > \mathrm{Re}(a)$$

5. $f(t) = \sin(at)$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

6. $f(t) = \cos(at)$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

7. $f(t) = e^{-bt}\sin(at)$

$$\mathcal{L}\{e^{-bt}\sin(at)\} = \frac{a}{(s+b)^2+a^2}$$

8. $f(t) = \delta(t)$ (Dirac Delta Function)

$$\mathcal{L}\{\delta(t)\} = 1$$

9. $f(t) = u(t-a)$ (Unit Step Function)

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

10. **Derivative Example:** Let $f(t) = e^{2t} \Rightarrow f'(t) = 2e^{2t},\ f(0) = 1$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) = s\left(\frac{1}{s-2}\right) - 1 = \frac{s}{s-2} - 1$$

11. **Derivative Example:** Let $f(t) = e^{2t} \Rightarrow f'(t) = 2e^{2t},\ f(0) = 1$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) = s\left(\frac{1}{s-2}\right) - 1 = \frac{s}{s-2} - 1$$

12. $f(t) = \sinh(at)$

$$\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2 - a^2}$$

13. $f(t) = \cosh(at)$

$$\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2 - a^2}$$

14. $f(t) = te^{at}$

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

15. $f(t) = t\sin(at)$

$$\mathcal{L}\{t\sin(at)\} = \frac{2as}{(s^2+a^2)^2}$$

16. $f(t) = t\cos(at)$

$$\mathcal{L}\{t\cos(at)\} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

17. $f(t) = (t-a)u(t-a)$

$$\mathcal{L}\{(t-a)u(t-a)\} = \frac{e^{-as}}{s^2}$$

18. $f(t) = u(t-2)\cdot\sin(t-2)$

$$\mathcal{L}\{u(t-2)\sin(t-2)\} = e^{-2s}\cdot\frac{1}{s^2+1}$$

19. $f(t) = \frac{\sin(at)}{t}$

$$\mathcal{L}\left\{\frac{\sin(at)}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right)$$

20. $f(t) = \frac{1-\cos(at)}{t}$

$$\mathcal{L}\left\{\frac{1-\cos(at)}{t}\right\} = \ln\left(\frac{s^2+a^2}{s^2}\right)$$

21. **Solve:** $f'(t) = \cos(2t), \ \ f(0) = 3$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) = \frac{s}{s^2+4}$$

$$sF(s) - 3 = \frac{s}{s^2+4} \Rightarrow F(s) = \frac{1}{s^2+4} + \frac{3}{s}$$

$$\Rightarrow f(t) = \frac{1}{2}\sin(2t) + 3$$

21. $f(t) = \int_0^t \tau\cos(a\tau)\,d\tau$

$$\mathcal{L}\left\{\int_0^t \tau\cos(a\tau)\,d\tau\right\} = \frac{1}{s}\cdot\mathcal{L}\{t\cos(at)\} = \frac{1}{s}\cdot\frac{s^2-a^2}{(s^2+a^2)^2}$$

22. $f(t) = e^{2t}\cos(3t)$

$$\mathcal{L}\{e^{2t}\cos(3t)\} = \frac{s-2}{(s-2)^2+9}$$

23. $f(t) = t^2 e^{at}$

$$\mathcal{L}\{t^2 e^{at}\} = \frac{2}{(s-a)^3}$$

24. **Inverse Laplace of** $F(s) = \dfrac{1}{(s+1)^2}$

$$f(t) = te^{-t}$$

25. $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 4-t, & t \ge 2 \end{cases}$

$$f(t) = t\cdot u(t) + (4-2t)\cdot u(t-2)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{2e^{-2s}}{s^2}$$

26. $f(t) = t * \sin(t)$

$$\mathcal{L}\{t * \sin(t)\} = \mathcal{L}\{t\}\cdot\mathcal{L}\{\sin(t)\} = \frac{1}{s^2}\cdot\frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)}$$

27. **Inverse Laplace of** $F(s) = \dfrac{s}{(s^2+4)^2}$

$$f(t) = t\cos(2t)$$

28. $f(t) = \begin{cases} t, & 0 < t < 2 \\ \textbf{repeats every } 2 \end{cases}$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 te^{-st}dt$$

$$\int_0^2 te^{-st}dt = \left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^2 = -\frac{2e^{-2s}}{s} - \frac{e^{-2s}-1}{s^2}$$

$$F(s) = \frac{1}{1-e^{-2s}}\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}-1}{s^2}\right)$$

Inverse Laplace Transform – Solved Examples

1. $\mathcal{L}^{-1}\left\{\dfrac{1}{s-5}\right\} = e^{5t}$

2. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+16}\right\} = \cos(4t)$

3. $\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\} = \sin(5t)$

4. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

5. $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^3}\right\} = \frac{t^2}{2}e^{2t}$

6. $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+2}\right\} = e^{-t}\cos(t)$

7. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\} = e^{-3t}\cdot\frac{\sin(2t)}{2}$

8. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+9}\right\} = e^{-t}\cdot\frac{\sin(3t)}{3}$

9. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = t\cos(3t)$

10. $\mathcal{L}^{-1}\left\{\frac{3s+5}{(s+2)^2+9}\right\} = 3e^{-2t}\cos(3t) + \frac{5}{3}e^{-2t}\sin(3t)$

11. **Find** $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+13}\right\}$

    Solution:

$$s^2+4s+13 = (s+2)^2+3^2$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+3^2}\right\} = \frac{1}{3}e^{-2t}\sin(3t)$$

12. **Find** $\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+4s+5}\right\}$

    Solution:

$$s^2+4s+5 = (s+2)^2+1, \quad 2s+4 = 2(s+2)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+1}\right\} = 2e^{-2t}\cos(t)$$

13. **Find** $\mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\}$

    Solution: Use partial fractions:

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$1 = A(s+3) + B(s) \Rightarrow A = \frac{1}{3}, \quad B = -\frac{1}{3}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{3s} - \frac{1}{3(s+3)}\right\} = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

14. **Find** $\mathcal{L}^{-1}\left\{\frac{5}{(s+1)^2}\right\}$

Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2}\right\} = te^{-at} \Rightarrow f(t) = 5te^{-t}$$

15. **Find** $\mathcal{L}^{-1}\left\{\frac{3s+6}{(s+3)^2+16}\right\}$

Solution:

$$\frac{3s+6}{(s+3)^2+4^2} = \frac{3(s+3)}{(s+3)^2+4^2} \Rightarrow f(t) = 3e^{-3t}\cos(4t)$$

Solving Differential Equations using Laplace Transforms

Example 1

Solve:

$$y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Step 1: Take Laplace transform

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

$$s^2Y(s) - s + 4Y(s) = 0$$

$$(s^2+4)Y(s) = s$$

Step 2: Solve for $Y(s)$

$$Y(s) = \frac{s}{s^2+4}$$

Step 3: Take inverse Laplace

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos(2t)$$

Final Answer:

$$\boxed{y(t) = \cos(2t)}$$

Example 2

Solve:

$$y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Step 1: Take Laplace transform

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$

$$s^2Y(s) - s + 3(sY(s) - 1) + 2Y(s) = 0$$

$$(s^2+3s+2)Y(s) = s+3$$

Step 2: Solve for $Y(s)$

$$Y(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$$

Step 3: Partial fractions

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$s+3 = A(s+2) + B(s+1) \Rightarrow A = 2, \quad B = -1$$

Step 4: Inverse Laplace

$$Y(s) = \frac{2}{s+1} - \frac{1}{s+2} \Rightarrow y(t) = 2e^{-t} - e^{-2t}$$

Final Answer:

$$\boxed{y(t) = 2e^{-t} - e^{-2t}}$$

3. **Solve:** $y'' + y = \sin t, \quad y(0) = 0, \quad y'(0) = 0$

   Solution:

$$s^2Y(s) + Y(s) = \frac{1}{s^2+1} \Rightarrow Y(s)(s^2+1) = \frac{1}{s^2+1} \Rightarrow Y(s) = \frac{1}{(s^2+1)^2}$$

$$\Rightarrow y(t) = t \sin t$$

4. **Solve:** $y'' - y = e^{2t}, \quad y(0) = 1, \quad y'(0) = 0$

   Solution:

$$s^2Y(s) - s - 1 - Y(s) = \frac{1}{s-2} \Rightarrow (s^2-1)Y(s) = \frac{1}{s-2} + s + 1$$

   (Requires algebraic simplification for partial fractions.)

5. **Solve:** $y'' + 2y' + y = \delta(t-3), \quad y(0) = 0, \quad y'(0) = 0$

   Solution:

$$(s+1)^2Y(s) = e^{-3s} \Rightarrow Y(s) = \frac{e^{-3s}}{(s+1)^2} \Rightarrow y(t) = (t-3)e^{-(t-3)}u(t-3)$$

6. **Solve:** $y' + 2y = 4, \quad y(0) = 1$

   Solution:

$$sY(s) - 1 + 2Y(s) = \frac{4}{s} \Rightarrow Y(s)(s+2) = \frac{4}{s} + 1 \Rightarrow Y(s) = \frac{4}{s(s+2)} + \frac{1}{s+2}$$

$$\frac{4}{s(s+2)} = \frac{2}{s} - \frac{2}{s+2} \Rightarrow y(t) = 2 - e^{-2t}$$

7. **Solve:** $y'' + 2y = \cos t, \quad y(0) = 1, \quad y'(0) = 0$

Solution:

$$s^2Y(s) - s + 2Y(s) = \frac{s}{s^2+1} \Rightarrow (s^2+2)Y(s) = s + \frac{s}{s^2+1}$$

(Requires further algebraic manipulation.)

8. **Solve:** $y'' + 2y' + 2y = \sin t, \quad y(0) = 0, \quad y'(0) = 1$

Solution:

$$s^2Y(s) - 1 + 2sY(s) + 2Y(s) = \frac{1}{s^2+1} \Rightarrow (s^2+2s+2)Y(s) = \frac{1}{s^2+1} + 1$$

(Continue with simplification and inverse Laplace.)

9. **Solve:** $y' + y = u(t-2), \quad y(0) = 0$

Solution:

$$sY(s) + Y(s) = \frac{e^{-2s}}{s} \Rightarrow Y(s) = \frac{e^{-2s}}{s(s+1)} \Rightarrow y(t) = (1 - e^{-(t-2)})u(t-2)$$

10. **Solve:** $y'' + 4y = \cos(2t), \quad y(0) = 0, \quad y'(0) = 0$

Solution:

$$s^2Y(s) + 4Y(s) = \frac{s}{s^2+4} \Rightarrow Y(s) = \frac{s}{(s^2+4)^2} \Rightarrow y(t) = \frac{1}{4}\sin(2t) - \frac{t}{2}\cos(2t)$$

11. **Solve:** $y'' + y = u(t-\pi), \quad y(0) = 0, \quad y'(0) = 0$

Solution:

$$(s^2+1)Y(s) = \frac{e^{-\pi s}}{s} \Rightarrow Y(s) = \frac{e^{-\pi s}}{s(s^2+1)}$$

$$y(t) = \int_0^{t-\pi} \sin(\tau)\, d\tau \cdot u(t-\pi) = [1 - \cos(t-\pi)]u(t-\pi)$$

12. **Solve:** $y'' + 2y' + 2y = \cos t, \quad y(0) = 0, \quad y'(0) = 1$

Solution:

$$s^2Y(s) - 1 + 2sY(s) + 2Y(s) = \frac{s}{s^2+1} \Rightarrow (s^2+2s+2)Y(s) = \frac{s}{s^2+1} + 1$$

Solve by partial fractions and take inverse Laplace transform.

13. **Solve:** $y'' + 4y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = -1$

Solution:

$$s^2Y(s) - 2s - 1 + 4sY(s) - 8 + 5Y(s) = 0 \Rightarrow (s^2+4s+5)Y(s) = 2s - 1 \Rightarrow Y(s) = \frac{2s-1}{(s+2)^2+1}$$

$$\Rightarrow y(t) = 2e^{-2t}\cos t - e^{-2t}\sin t$$

14. **Solve:** $y'' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$

Solution:

$$(s^2+2)Y(s) = \frac{1}{s+1} \Rightarrow Y(s) = \frac{1}{(s+1)(s^2+2)}$$

Use partial fractions and inverse Laplace.

15. **Solve:** $y' + y = \cos t, \quad y(0) = 1$

Solution:

$$sY(s) - 1 + Y(s) = \frac{s}{s^2+1} \Rightarrow Y(s)(s+1) = \frac{s}{s^2+1} + 1 \Rightarrow Y(s) = \frac{s}{(s+1)(s^2+1)} + \frac{1}{s+1}$$

Use partial fractions and inverse Laplace.

16. **Solve:** $y'' + 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 1$

Solution:

$$s^2Y(s) - 1 + 6sY(s) + 9Y(s) = 0 \Rightarrow (s+3)^2Y(s) = 1 \Rightarrow Y(s) = \frac{1}{(s+3)^2} \Rightarrow y(t) = te^{-3t}$$

17. **Solve:** $y'' = \delta(t-2), \quad y(0) = 0, \quad y'(0) = 0$

Solution:

$$s^2Y(s) = e^{-2s} \Rightarrow Y(s) = \frac{e^{-2s}}{s^2} \Rightarrow y(t) = (t-2)u(t-2)$$

18. **Solve:** $y' + 3y = 5e^{-2t}, \quad y(0) = 2$

Solution:

$$sY(s) - 2 + 3Y(s) = \frac{5}{s+2} \Rightarrow Y(s)(s+3) = \frac{5}{s+2} + 2 \Rightarrow Y(s) = \frac{5}{(s+3)(s+2)} + \frac{2}{s+3}$$

19. **Solve:** $y'' + y = u\left(t - \frac{\pi}{2}\right), \quad y(0) = 0, \quad y'(0) = 0$

Solution:

$$(s^2+1)Y(s) = \frac{e^{-\frac{\pi s}{2}}}{s} \Rightarrow Y(s) = \frac{e^{-\frac{\pi s}{2}}}{s(s^2+1)}$$

$$\Rightarrow y(t) = [1 - \cos(t - \frac{\pi}{2})]\, u\left(t - \frac{\pi}{2}\right)$$

20. **Solve:** $y'' - 2y' + y = te^t, \quad y(0) = 0, \quad y'(0) = 1$

Solution:

$$s^2Y(s) - s - 1 - 2sY(s) + Y(s) = \frac{1}{(s-1)^2} \Rightarrow (s^2 - 2s + 1)Y(s) = \frac{1}{(s-1)^2} + s + 1$$

Example 21: Solve:

$$y'' + 4y = \sin(2t)u(t-\pi), \quad y(0) = 0, \quad y'(0) = 0$$

Solution: Laplace of RHS:

$$\mathcal{L}\{\sin(2(t-\pi))u(t-\pi)\} = e^{-\pi s} \cdot \frac{2}{s^2+4}$$

LHS:

$$(s^2+4)Y(s) = e^{-\pi s} \cdot \frac{2}{s^2+4} \Rightarrow Y(s) = e^{-\pi s} \cdot \frac{2}{(s^2+4)^2}$$

$$y(t) = u(t-\pi) \cdot (t-\pi)\sin(2(t-\pi))$$

Example 22: Solve:

$$y'' + 6y' + 8y = \delta(t-3), \quad y(0) = 0, \quad y'(0) = 0$$

Solution: Laplace of RHS: $\mathcal{L}\{\delta(t-3)\} = e^{-3s}$

$$(s^2+6s+8)Y(s) = e^{-3s} \Rightarrow Y(s) = \frac{e^{-3s}}{(s+2)(s+4)}$$

Use partial fractions and inverse Laplace:

$$y(t) = u(t-3)[Ae^{-2(t-3)} + Be^{-4(t-3)}]$$

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Example 23: Solve:

$$y'' - 2y' + y = e^{3t}, \quad y(0) = 2, \quad y'(0) = 0$$

Solution: Laplace of RHS: $\frac{1}{s-3}$

$$(s^2-2s+1)Y(s) - 2s + 0 + 4 = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{(s-3)(s-1)^2} + \text{initial condition terms}$$

Use partial fractions.

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Example 24: Solve:

$$y' + y = u(t-1), \quad y(0) = 0$$

Solution: Laplace of RHS: $\frac{e^{-s}}{s}$

$$sY(s) + Y(s) = \frac{e^{-s}}{s} \Rightarrow Y(s) = \frac{e^{-s}}{s(s+1)}$$

Inverse:

$$y(t) = u(t-1)[1 - e^{-(t-1)}]$$

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Example 25: Solve:

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 1$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t < 2 \\ 0, & t \geq 2 \end{cases} = 1 - u(t-2)$$

Solution:

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$(s^2+1)Y(s) - 1 = \frac{1 - e^{-2s}}{s} \Rightarrow Y(s) = \frac{1}{s^2+1}\left[\frac{1-e^{-2s}}{s} + 1\right]$$

Inverse involves convolution and unit step handling.