Vibrations Summary

1. Free Vibrations

1.1 Introduction to Vibrations

Vibrations are often unwanted phenomena in aerospace engineering. When systems start vibrating at the wrong frequencies, they might fail, which isn't particulary good. In reality all systems are **continuous systems**, meaning that the displacements of parts depend on a lot of factors. To simplify this, the system is often modeled as a **discrete system**. Here the system is split up in parts, which are then evaluated separately.

Two types of vibrations can be distinguished, being **free vibrations** and **forced vibrations**. In free vibrations no energy is exchanged with the environment, while in forced vibrations there is energy exchange. First we will have a look at free vibrations. Forced vibrations will be treated in later chapters.

1.2 Stiffness of an Axially Loaded Rod

Let's consider an axially loaded rod of negligible mass, having a mass attached to its end. We know that the displacement δ of the mass is given by

$$\delta = \frac{FL}{EA}, \tag{1.2.1}$$

where F is the (tensional) force in the bar, L is the length of the bar, E is the E-modulus and A is the cross-sectional area. The **stiffness** k is defined as the force needed to reach unit displacement. In an equation this is

$$k = \frac{F}{\delta}. \tag{1.2.2}$$

So for our axially loaded rod, we will have

$$k = \frac{EA}{L}. \tag{1.2.3}$$

We can now model the situation. We do this by replacing the bar by a spring with the stiffness k. This is shown in figure 1.1.

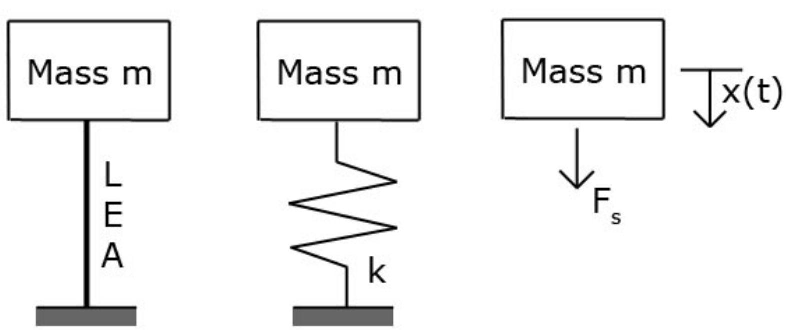

Figure 1.1: Modeling of an axially loaded rod.

1.3 Motion of an Axially Loaded Rod

Previously we considered the axially loaded rod and modeled it. Let's turn to figure 1.1 once more. We would like to know how the system will move, if it is given a certain initial displacement/velocity.

To find this out, we use Newton's second law $F = ma$. The only force acting on the mass is the spring force F_s (we don't consider gravity yet). We know that the spring force varies linearly with the displacement x by the stiffness k. However, if the block moves upward, the spring forces points downward. So there is a negative relation between the two. In an equation this becomes

$$F_s = -kx. \tag{1.3.1}$$

If we combine this with Newton's second law, we will find that

$$m\ddot{x} = F_s = -kx \qquad \Rightarrow \qquad m\ddot{x} + kx = 0. \tag{1.3.2}$$

The solution can be found by solving this differential equation. We will get

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right). \tag{1.3.3}$$

So the system will start vibrating with a fixed angular frequency. This frequency, called the **angular eigenfrequency**, is denoted by

$$\omega_n = \sqrt{\frac{k}{m}}. \tag{1.3.4}$$

From this, the **eigenfrequency** f and **vibration period** T can be derived, according to

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{and} \qquad T = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}. \tag{1.3.5}$$

However, equation 1.3.3 isn't very useful. Instead, it is more meaningful to use

$$x(t) = A\sin\left(\omega_n t + \phi\right), \tag{1.3.6}$$

where A is the amplitude (usually taken to be positive) and ϕ is the phase. Both follow from the boundary conditions. If we give the mass an initial displacement x_0 and an initial velocity v_0, then we can find A and ϕ. They will be

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right). \tag{1.3.7}$$

1.4 Effects of Gravity

Previously we haven't considered gravity. What happens if we do? In this case the total force acting on the mass will be $F_s + mg$. This would turn the differential equation into

$$m\ddot{x} + kx = mg. \tag{1.4.1}$$

When solving differential equations, we know we first ought to find the **homogeneous solution** of the differential equation

$$m\ddot{x} + kx = 0. \tag{1.4.2}$$

We already know the solution for this. After we have found the homogeneous solution, we need to find one **particular solution** $x_p(t)$. Note that the non-homogeneous term mg is just a constant. So the particular solution is probably constant too. It can then be shown that

$$x_p(t) = \frac{mg}{k}. \tag{1.4.3}$$

This makes the solution for the differential equation

$$x(t) = x_h(t) + x_p(t) = A\sin(\omega_n t + \phi) + \frac{mg}{k}. \tag{1.4.4}$$

Note that if the amplitude A is zero, then the mass will just have a constant displacement of mg/k. This also follows from statics.

In vibrational engineering the homogeneous solution $x_h(t)$ is sometimes called the **transient solution** $x_{tr}(t)$ and the particular solution $x_p(t)$ is also called the **steady state solution** $x_{ss}(t)$.

1.5 Motion of a Laterally Loaded Rod

Of course there are more kinds of vibrations then masses on axially loaded rods. Let's consider a laterally loaded rod, as shown in figure 1.2. The rod has an (area) moment of inertia I.

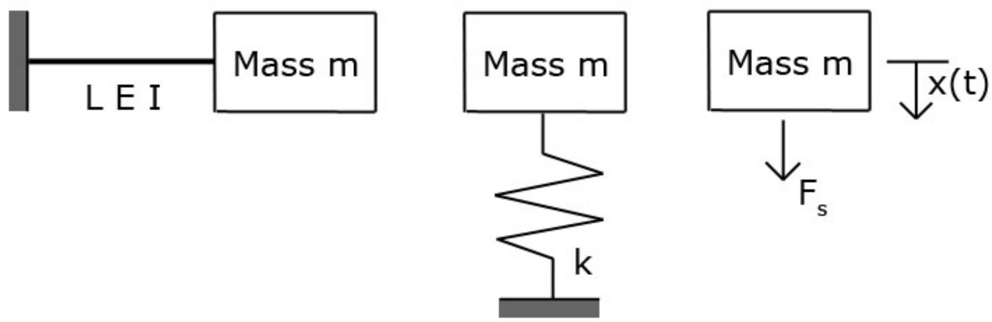

Figure 1.2: Modeling of a laterally loaded rod.

This time the displacement δ, and thus also the stiffness k and natural frequency ω_n, are given by

$$\delta = \frac{FL^3}{3EI} \qquad \Rightarrow \qquad k = \frac{F}{\delta} = \frac{3EI}{L^3} \qquad \Rightarrow \qquad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{mL^3}}. \tag{1.5.1}$$

The rest of the problem is similar to what we have previously discussed.

1.6 Rotation of a Torsionally Loaded Rod

Now let's consider an other case. We have a disk with (mass) moment of inertia J, connected to a rod with (area) polar moment of inertia I_p, as shown in figure 1.3.

We will be looking at the angular displacement θ. This depends on the moment M that is acting between the rod and the disk. If this moment is known, then the angular displacement can be found using

$$\theta = \frac{ML}{GI_p}. \tag{1.6.1}$$

Now we can define the **torsional stiffness** as

$$k = -\frac{M}{\theta} = \frac{GI_p}{L}. \tag{1.6.2}$$

Note that the torsional stiffness has as unit Nm, while the normal stiffness has as unit N/m.

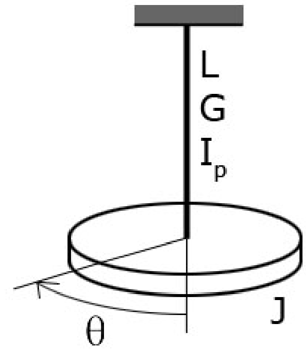

Figure 1.3: Modeling of a torsionally loaded rod.

Newton's second law for rotations states that $M = J\alpha = J\ddot{\theta}$. Combining this with the torsional stiffness gives us the differential equation

$$J\ddot{\theta} + k\theta = 0. \tag{1.6.3}$$

We already know the solution to this! It is just

$$\theta(t) = \hat{\theta}\sin\left(\omega_n t + \phi\right), \tag{1.6.4}$$

where $\omega_n = \sqrt{k/J}$ is the angular natural frequency and $\hat{\theta}$ denotes the amplitude of the vibration.

1.7 Other Cases

We have seen axially loaded rods, laterally loaded rods and torsionally loaded rods. There are, however, infinitely many other types of systems. It is, for example, possible to combine multiple springs in a system. We won't be treating all those combinations, of course. If this is the case, the skills of the engineer come into play.

However, we're not letting you venture into those problems unguided. When face with a more complicated system, just follow the following steps:

- Consider the point of which you want to know the motion.
- Express the force/moment at that point as a function of the (angular) displacement.
- Use Newton's second law to find the differential equation.
- Solve the differential equation to find the equation of motion.

1.8 Using Energy

In a free vibration (without damping), energy is conserved. You can consider two types of energy in a vibration. These are **kinetic energy** T and **potential energy** U. Let's consider those energies for the axially/laterally loaded rod. The kinetic energy is given by

$$T = \frac{1}{2}m\dot{x}^2. \tag{1.8.1}$$

The potential energy here consists of spring energy and gravitational energy, and is given by

$$U = \frac{1}{2}kx^2 - mgx, \tag{1.8.2}$$

A very important rule is the rule of **conservation of energy**. It states that

$$T + U = \text{constant} = E, \tag{1.8.3}$$

where E is the **vibrational energy**. If the mass passes through the equilibrium point, then T is maximal. If the mass has maximum deflection, then U is maximal.

It all sounds fun, but how can we use this? To use this, we differentiate equation 1.8.3 with respect to time. What we get is

$$\frac{dT}{dt} + \frac{dU}{dt} = 0. \tag{1.8.4}$$

If we work this out for an axially/laterally loaded rod, we will get

$$\dot{x}\left(m\ddot{x} + kx - mg\right) = 0. \tag{1.8.5}$$

Note that $\dot{x}$ can't be zero for all t (or it would be an awfully boring problem). We now remain with

$$m\ddot{x} + kx = mg, \tag{1.8.6}$$

which is exactly the differential equation we needed to solve the problem.

The method that was just shown is called the **energy method**. When damping occurs, the energy method is slightly more complicated. Now the lost energy also needs to be taken into account. We will not treat this here though.

You may be wondering why we should use energy? Isn't it easier to just use Newton's second law? Well, using Newton's second law is easier for normal one-dimensional problems. However, using energy when solving multi-dimensional problems has various advantages. We will consider multi-dimensional problems in more detail in the latest chapter of this summary.

2. Damped Motions

2.1 Introduction to Damping

The free vibrations discussed in the previous chapter don't stop oscillating. This isn't very realistic. So we need to change our model. We therefore apply **viscous damping**. We assume that there is a force acting on the mass in a direction opposite to the motion. This force is also proportional to the motion (fast-moving objects have more friction). So we introduce the **damping force**

$$f_c = -c\dot{x}(t), \tag{2.1.1}$$

where the factor $c > 0$ is the **damping coefficient**. If we combine this with the previous differential equation, we now get

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{2.1.2}$$

To solve this differential equation, we should first solve the **characteristic equation**

$$m\lambda^2 + c\lambda + k = 0 \qquad \Rightarrow \qquad \lambda = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}. \tag{2.1.3}$$

The behaviour of the system now depends on the factor $c^2 - 4km$. Different things occur if this factor is either smaller than zero, equal to zero or bigger than zero. Since this is so important, the **critical damping coefficient** c_{cr} is defined such that

$$c_{cr}^2 - 4km = 0 \qquad \Rightarrow \qquad c_{cr} = 2\sqrt{km} = 2m\omega_n. \tag{2.1.4}$$

Here ω_n is the natural frequency of the undamped system, also called the **undamped natural frequency**. We can now also define the **damping ratio** as

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}. \tag{2.1.5}$$

Note that since $c > 0$ also $\zeta > 0$. Using ζ, the characteristic equation can be rewritten as

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \tag{2.1.6}$$

Three cases can now be distinguished, which will be treated in the coming paragraphs.

2.2 Underdamped Motion

In the **underdamped motion** the damping ratio ζ is smaller than one. The solutions λ_1 and λ_2 of the characteristic equation are now complex conjugates, being

$$\lambda_1 = -\zeta\omega_n - \omega_n\sqrt{1-\zeta^2}\,i \qquad \text{and} \qquad \lambda_2 = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}\,i. \tag{2.2.1}$$

Before we write down the solution, we first define the **damped natural frequency** to be

$$\omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{2.2.2}$$

If we now solve the differential equation, we will find as the general solution

$$x(t) = Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right), \tag{2.2.3}$$

where A is the **initial amplitude**. Note that due to damping, the frequency of the vibration has changed. The values of A and ϕ depend on the initial position x_0 and initial velocity v_0 and can be found using

$$A = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right). \tag{2.2.4}$$

The underdamped motion results in an oscillation with a decreasing amplitude.

2.3 Overdamped Motion

In the **overdamped motion** the damping ratio ζ is bigger than one. The roots to the characteristic equation are now two real values, being

$$\lambda_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \qquad \text{and} \qquad \lambda_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}. \tag{2.3.1}$$

In this case no oscillation occurs. The mass will not even pass the equilibrium position. Instead, it will only converge to it. Before we see how, we first define

$$\omega_c = \omega_n\sqrt{\zeta^2 - 1}. \tag{2.3.2}$$

The motion of the mass is now described by

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_c t} + a_2 e^{\omega_c t}\right). \tag{2.3.3}$$

The constants a_1 and a_2 once more depend on the initial conditions. They can be found using

$$a_1 = \frac{1}{2}x_0\left(1 - \frac{\zeta\omega_n}{\omega_c}\right) - \frac{v_0}{2\omega_c} \qquad \text{and} \qquad a_2 = \frac{1}{2}x_0\left(1 + \frac{\zeta\omega_n}{\omega_c}\right) + \frac{v_0}{2\omega_c}. \tag{2.3.4}$$

2.4 Critically Damped Motion

In the **critically damped motion** we have $\zeta = 1$ and thus $c = c_{cr}$. The roots of the characteristic equation are now

$$\lambda_1 = \lambda_2 = -\omega_n. \tag{2.4.1}$$

The solution is now given by

$$x(t) = (a_1 + a_2 t)\, e^{-\omega_n t}, \tag{2.4.2}$$

where the constants a_1 and a_2 are given by

$$a_1 = x_0 \qquad \text{and} \qquad a_2 = v_0 + \omega_n x_0. \tag{2.4.3}$$

2.5 Stability

We have, up to now, considered only positive k and c. Of course it is also possible to have a negative k (the force acts in the direction of the displacement or a negative c (the force acts in the direction of motion.

If, for a certain motion, $x \to \infty$, then the motion is **unstable**. Otherwise the motion is **stable**. We will look at the stability of the systems for various combinations of c and k now.

- $k > 0$ - This occurs in normal springs. In case of a deflection, the mass is pulled back to the equilibrium position.
  - For $c = 0$ we are on familiar grounds. The motion is just an undamped vibration. The amplitude is bounded ($x(t) \leq A$ for all t) so we have a stable motion. However, $x(t)$ never converges to zero. So the system is only **marginally stable**.
  - For $c > 0$ we are dealing with a damped motion. It doesn't matter whether the system is underdamped, overdamped or critically damped. In all cases $x(t) \to 0$ as $t \to \infty$, so the system is asymptotically stable. (How x goes to zero does depend on ζ though, but this is irrelevant for the stability.)
  - If $c < 0$, then the amplitude of the motion increases unbounded for increasing t. So the motion is **unstable**. However, we can distinguish two cases.

* $*$ If $c^2 < 4mk$ (thus $\zeta < 1$), then there are still oscillations. In this case we have **flutter instability**.
* $*$ For $c^2 \geq 4mk$ (thus $\zeta \geq 1$) no oscillation occurs. As soon as the mass departs from the equilibrium, it will never return. Now there is **divergent instability**.

- When $k < 0$ the mass gets pushed away from the equilibrium position, independent of the damping coefficient c. For $c > 0$ the motion only occurs slower than for $c < 0$. Since $x(t) \to \infty$ as $t \to \infty$, the motion is **unstable**. To be more precise, there is **divergent instability**, since not a single oscillation occurs.

2.6 Coulomb Friction

Suppose we have mass, horizontally sliding over a surface, as shown in figure 2.4.

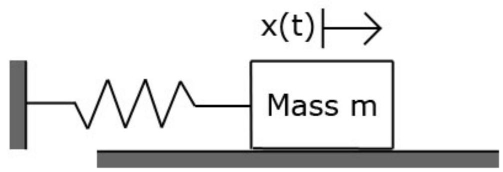

Figure 2.4: Mass connected to a spring, sliding over a horizontal surface.

The force that acts on the mass depends on whether it is moving, and in which direction, according to

$$f_c(\dot{x}) = \begin{Bmatrix} -\mu N & \text{if } \dot{x} > 0 \\ 0 & \text{if } \dot{x} = 0 \\ \mu N & \text{if } \dot{x} < 0 \end{Bmatrix} = \mu N \begin{Bmatrix} -1 & \text{if } \dot{x} > 0 \\ 0 & \text{if } \dot{x} = 0 \\ 1 & \text{if } \dot{x} < 0 \end{Bmatrix} = -\mu N \text{sgn}(\dot{x}), \tag{2.6.1}$$

where μ is the **dynamic friction coefficient** and N is the normal force acting on the block. Also $\text{sgn}(\tau)$ is the **signum function**, defined to give 1 when $\tau > 0$, 0 when $\tau = 0$ and -1 if $\tau < 0$. This kind of damping is called **Coulomb damping**. The resulting differential equation is

$$m\ddot{x} + \mu N \text{sgn}(\dot{x}) + kx = 0. \tag{2.6.2}$$

This is very hard to solve, due to the signum function. It is wiser to examine the problem in steps. Suppose the mass has no initial velocity ($v_0 = 0$), but only an initial displacement δ_0. If the initial displacement is big enough to overcome the friction force ($k\delta_0 > \mu N$), the block will start sliding. After π/ω_n seconds it will have reached a new maximum deflection δ_1. It can be shown that this deflection is

$$\delta_1 = \delta_0 - \frac{2\mu N}{k}. \tag{2.6.3}$$

If the force is big enough to let the block slide again, it will have another half oscillation of π/ω_n seconds, but its maximum deflection will have decreased again by the same amount. So,

$$\delta_2 = \delta_1 - \frac{2\mu N}{k}. \tag{2.6.4}$$

This continues until after i half oscillations $k\delta_i \leq \mu N$. The block has been oscillating for $i\pi/\omega_n$ seconds. But now the oscillation has ended and the block will remain at δ_i.

3. Harmonic Excitation

3.1 Introduction to Harmonic Excitation

In the previous chapters, the only force present was the force of the spring. Although we also considered gravity, this was a constant force and thus not very interesting. What will happen if we cause a time-dependent **external force** $F_e(t)$ on the mass? In this case the differential equation for an undamped motion should be rewritten to

$$m\ddot{x} + kx = F_e(t). \tag{3.1.1}$$

We can get about any motion, depending on the external force. In reality external forces are often harmonic. We therefore assume that

$$F_e(t) = \hat{F}_e \cos \omega t, \tag{3.1.2}$$

where ω is the **angular frequency of the external force**. To solve this differential equation, we first need to find the homogeneous solution. This solution is already known from previous chapters though. So we focus on the particular solution $x_p(t)$. We assume that it can be written as

$$x_p(t) = \hat{x}_p \cos \omega t. \tag{3.1.3}$$

Inserting this in the differential equation will give

$$\hat{x}_p = \frac{\hat{F}_e}{m} \frac{1}{(\omega_n^2 - \omega^2)} \qquad \Rightarrow \qquad x_p(t) = \frac{\hat{F}_e}{m} \frac{1}{(\omega_n^2 - \omega^2)} \cos \omega t. \tag{3.1.4}$$

If we combine this with the general solution to the homogeneous problem, we find that

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t + \frac{\hat{F}_e}{m} \frac{1}{(\omega_n^2 - \omega^2)} \left(\cos \omega t - \cos \omega_n t\right). \tag{3.1.5}$$

A very important thing can be noticed from this equation. If $\omega \to \omega_n$, then $x_p(t) \to \infty$ and thus also $x(t) \to \infty$. This phenomenon is called **resonance** and is defined to occur if $\omega = \omega_n$. It is something engineers should definitely prevent.

3.2 Resonance

When looking at equation 3.1.5 we can see that it is undefined for $\omega = \omega_n$. What happens if we force a system to vibrate at its natural frequency? To find this out, we set $\omega = \omega_n$. The differential equation now becomes

$$\ddot{x} + \omega_n^2 x(t) = \frac{\hat{F}_e}{m} \cos \omega_n t. \tag{3.2.1}$$

If we try a solution of the form $x_p(t) = \hat{x}_p \cos \omega_n t$, we will only find the equation $0 = \left(\hat{F}_e/m\right) \cos \omega_n t$. So there are no solutions of the assumed form. Instead, let's try to assume that $x_p(t) = \hat{x}_p t \sin \omega_n t$. We now find that

$$\hat{x}_p = \frac{\hat{F}_e}{2m\omega_n} \qquad \Rightarrow \qquad x_p(t) = \frac{\hat{F}_e}{2m\omega_n} t \sin \omega_n t. \tag{3.2.2}$$

What we get is a vibration in which the amplitude increases linearly with time. So as the time t increases, also the amplitude of the motion increases. This continues until the system can't sustain the large amplitudes anymore and will fail.

3.3 Beat Phenomenon

When the external force isn't vibrating at exactly the natural frequency of a system, but only close to it, also interesting things occur. First let's define the two variables $\Delta\omega$ and $\bar{\omega}$ as

$$\Delta\omega = \frac{\omega_n - \omega}{2} \qquad \text{and} \qquad \bar{\omega} = \frac{\omega_n + \omega}{2}. \tag{3.3.1}$$

Let's once more consider equation 3.1.5. If we have no initial displacement or velocity ($x_0 = 0$ and $v_0 = 0$), then we can rewrite this equation to

$$2\frac{\hat{F}_e}{m}\frac{1}{(\omega_n^2 - \omega^2)}\sin(\Delta\omega t)\sin(\bar{\omega}t) = 2\frac{\hat{F}_e}{m}\frac{1}{(\omega_n^2 - \omega^2)}\sin\left(\frac{2\pi}{T_1}t\right)\sin\left(\frac{2\pi}{T_2}t\right). \tag{3.3.2}$$

As $\omega \to \omega_n$ also $\Delta\omega \to 0$ and $\bar{\omega} \to \omega_n$. So it follows that T_1 will become very large, while T_2 is close to the natural frequency of the system. Since T_1 is so large, we can define the amplitude of the vibration as

$$A(t) = 2\frac{\hat{F}_e}{m}\frac{1}{(\omega_n^2 - \omega^2)}\sin\left(\frac{2\pi}{T_1}t\right). \tag{3.3.3}$$

So we now have a rapid oscillation with a slowly varying amplitude. This phenomenon is called the **beat phenomenon** and one variation of the amplitude is called a **beat**. As the forcing frequency ω goes closer to the natural frequency ω_n, both the amplitude and the period of a beat increase.

3.4 Harmonic Excitation of Damped Systems

Let's involve damping in our equations. We then get

$$m\ddot{x} + c\dot{x} + kx = \hat{F}_e\cos\omega t \qquad \Leftrightarrow \qquad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{\hat{F}_e}{m}\cos\omega t. \tag{3.4.1}$$

Let's assume our particular solution can be written as

$$x_p(t) = X\cos(\omega t - \theta). \tag{3.4.2}$$

Inserting this in the differential equation, and solving for X and θ, will eventually give

$$X = \frac{\hat{F}_e}{m}\frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \qquad \text{and} \qquad \theta = \arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right). \tag{3.4.3}$$

To find the general solution set, add $x_p(t)$ up to the solution of the homogeneous equation and use initial conditions to solve for the coefficients A and ϕ.

Let's define the (dimensionless) **frequency ratio** as

$$r = \frac{\omega}{\omega_n}. \tag{3.4.4}$$

We can now rewrite X and θ to

$$X = \frac{\hat{F}_e}{k}\frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \text{and} \qquad \theta = \arctan\left(\frac{2\zeta r}{1 - r^2}\right). \tag{3.4.5}$$

If $r \to 1$ then X goes to a given maximum value. This maximum value strongly depends on the damping ratio ζ. For large values of ζ, resonance is hardly a problem. However, if ζ is small, resonance can still occur.

3.5 Sinusoidal Forcing Functions

We have up to know only considered forcing functions involving a cosine. Of course forcing functions can also be expressed using a sine. Let's examine the forcing function

$$F_e(x) = \hat{F}_e \sin \omega t. \tag{3.5.1}$$

The particular solution to the (damped) differential equation then becomes

$$x_p(t) = X \sin(\omega t - \theta). \tag{3.5.2}$$

The variables X and θ are still the same as in equation 3.4.5.

3.6 Base Excitation

Let's now suppose no external force is acting on the mass. Instead the base on which the spring is connected, is moving by an amount $x_b(t)$, as shown in figure 3.5.

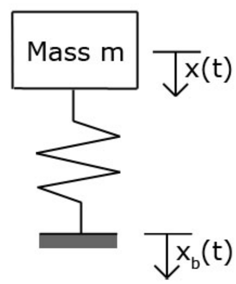

Figure 3.5: Definition of variables in base excitation.

The elongation of the spring is now not given by just $x(t)$, but by $x(t) - x_b(t)$. Identically, its velocity with respect to the ground is now $\dot{x}(t) - \dot{x}_b(t)$. So this makes the differential equation describing the problem

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{x}_b(t) + \omega_n^2 x_b(t). \tag{3.6.1}$$

Often the base excitation is harmonic, so we assume that

$$x_b(t) = \hat{x}_b \sin \omega_b t. \tag{3.6.2}$$

This makes the differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega_b\hat{x}_b \cos \omega_b t + \omega_n^2\hat{x}_b \sin \omega_b t. \tag{3.6.3}$$

We have two nonhomogeneous parts. We can therefore find two separate particular solutions for the differential equation (one for each part). If we set $\hat{F}_e/m = 2\zeta\omega_n\omega_b$ (or identically $\hat{F}_e/k = 2\zeta r$), then we have exactly the same problem as we have seen earlier with the cosine forcing function (equation 3.4.2). If we, on the other hand, set $\hat{F}_e/m = \omega_n^2\hat{x}_b$ (or identically $\hat{F}_e/k = \hat{x}_b$), then we have the same problem as we just saw with the sine forcing function (equation 3.5.2). Add the two solutions up to get the total particular solution

$$x_p(t) = \frac{2\zeta r\hat{x}_b}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} \cos(\omega t - \theta) + \frac{\hat{x}_b}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} \sin(\omega t - \theta). \tag{3.6.4}$$

The value of θ is still the same as it was in equation 3.4.5.

4. General Forced Vibrations

4.1 The Impulse Function

An **impulse excitation** is a force that is applied for a very short duration Δt with respect to the vibration period $T = 2\pi/\omega_n$. It is an example of a **shock loading**. Such an impulse can be mathematically represented by using the **unit impulse function** $\delta(t)$ (also called the **Dirac delta function**), defined such that

$$\delta(t-\tau) = 0 \qquad \text{for } t \neq \tau, \tag{4.1.1}$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)dt = 1. \tag{4.1.2}$$

But how does this effect the motion of a system? Let's suppose we have a system with no initial displacement and mass, that is given an impulse $\hat{F}_e$ at time $t = \tau$. The corresponding differential equation is

$$m\ddot{x} + c\dot{x} + kx = F_e\delta(t-\tau). \tag{4.1.3}$$

This impulse will cause the linear momentum of the mass to change by

$$\hat{F}_e = F_e\Delta t = m\,\Delta v = mv_\tau. \tag{4.1.4}$$

So this situation is similar to the case where the object simply has an initial velocity of v_τ at time $t = \tau$ (with $x_\tau = 0$). If we apply this, for example, to an underdamped system, we would get the equation of motion

$$x(t) = \hat{F}_e h(t-\tau), \qquad \text{where} \qquad h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t. \tag{4.1.5}$$

The function $h(t)$ is now called the **impulse response function**.

4.2 The Step Function

Another case of a forcing function is the **unit step function** $u(t)$ (also called the **Heaviside step function**, defined such that

$$u(t-\tau) = \begin{cases} 0 & \text{for } t < \tau, \\ 1 & \text{for } t \geq \tau. \end{cases} \tag{4.2.1}$$

Let's consider the underdamped differential equation

$$m\ddot{x} + c\dot{x} + kx = \hat{F}_e u(t-\tau). \tag{4.2.2}$$

If $x_0 = 0$ and $v_0 = 0$, it can be shown that

$$x(t) = \frac{\hat{F}_e}{k}\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \theta)\right), \tag{4.2.3}$$

where θ is given by

$$\theta = \arctan\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right). \tag{4.2.4}$$

This solution looks awfully familiar. In fact, it corresponds to a vibration with equilibrium point $x_e = \hat{F}_e/k$ and initial displacement $x_0 = 0$.

4.3 Replacing a Periodic Forcing Function by a Fourier Series

What if we don't have just an impulse or a step function, but a continuous forcing function $F_e(t)$? In this case we can take the force $F_e(\tau)$ at time τ for a given moment $d\tau$ and replace it by an impulse of magnitude $F_e(\tau)d\tau$. We can then find the impulse response function $h(t-\tau)$ for the time τ. If we do this for all times τ and sum everything up, we will eventually find as particular solution

$$x_p(t) = \int_0^t F_e(\tau)h(t-\tau)d\tau = \int_0^t F_e(t-\tau)h(\tau)d\tau. \tag{4.3.1}$$

This integral is called the **convolution integral**. It is often difficult to evaluate the integral. If we have a periodic forcing function $F_e(t)$ (with period T and angular frequency $\omega_T = 2\pi/T$), we can apply a trick though. We can replace $F_e(t)$ by a **Fourier series**. To do this, we use

$$F_e(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(n\frac{2\pi}{T}t\right) + b_n \sin\left(n\frac{2\pi}{T}t\right)\right). \tag{4.3.2}$$

The coefficients a_0, a_n and b_n are given by

$$a_0 = \frac{2}{T}\int_0^T F_e(t)dt, \tag{4.3.3}$$

$$a_n = \frac{2}{T}\int_0^T F_e(t)\cos\left(n\frac{2\pi}{T}t\right)dt, \tag{4.3.4}$$

$$b_n = \frac{2}{T}\int_0^T F_e(T)\sin\left(n\frac{2\pi}{T}t\right)dt. \tag{4.3.5}$$

Now we have a new way to write the forcing function. How we use this will be treated in the next paragraph.

4.4 Finding the Equation of Motion

When we replace the periodic forcing function $F_e(t)$ by a Fourier Series, we can rewrite the differential equation to

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(n\omega_T t\right) + b_n \sin\left(n\omega_T t\right)\right). \tag{4.4.1}$$

We now repeatedly take one element from the right hand side of the equation, solve the equation for that part, and in the end sum everything up. We will then find our particular solution. In an equation this becomes

$$x_p(t) = x_{a_0}(t) + \sum_{n=1}^{\infty}\left(x_{a_n}(t) + x_{b_n}(t)\right). \tag{4.4.2}$$

The individual solution are then the solutions of the differential equations

$$m\ddot{x}_{a_0} + c\dot{x}_{a_0} + kx_{a_0} = a_0/2, \tag{4.4.3}$$

$$m\ddot{x}_{a_n} + c\dot{x}_{a_n} + kx_{a_n} = a_n \cos\left(n\omega_T t\right), \tag{4.4.4}$$

$$m\ddot{x}_{b_n} + c\dot{x}_{b_n} + kx_{b_n} = b_n \sin\left(n\omega_T t\right). \tag{4.4.5}$$

All these equations are equations we have solved before. For completeness' sake we will give the solutions once more. They are

$$x_{a_0} = \frac{a_0}{2k}, \tag{4.4.6}$$

$$x_{a_n} = \frac{a_n}{m} X \cos\left(n\omega_T t - \theta_n\right), \tag{4.4.7}$$

$$x_{b_n} = \frac{b_n}{m} X \sin\left(n\omega_T t - \theta_n\right). \tag{4.4.8}$$

The variables X and θ_n are defined as

$$X = \frac{1}{\sqrt{\left(\omega_n^2 - (n\omega_T)^2\right)^2 + (2\zeta n\omega_n\omega_T)^2}} \qquad \text{and} \qquad \theta_n = \arctan\left(\frac{2\zeta n\omega_n\omega_T}{\omega_n^2 - (n\omega_T)^2}\right). \tag{4.4.9}$$

This is how the particular solution is found. Combine this with the specific solution to the problem to find the general solution to the differential equation.

4.5 Using the Laplace Transform

When solving the differential equation, the **Laplace transform** is often a convenient tool. Let's consider the differential equation

$$m\ddot{x} + c\dot{x} + kx = F_e(x) \qquad \Leftrightarrow \qquad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_e(x)}{m}. \tag{4.5.1}$$

Taking the laplace transform, and solving for $X(s)$, will give

$$X(s) = \frac{sx_0 + v_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{1}{m}\frac{\mathcal{F}_e(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{4.5.2}$$

where $L\{F_e(t)\} = \mathcal{F}_e(s)$. Often it occurs that $x_0 = 0$ and $v_0 = 0$. The middle term of the above equation then disappears. To find $x(t)$, you apply the inverse Laplace transform. When doing this, you often need to use a Laplace transform table like table 4.1.

Function $x(t) = L^{-1}\{X(s)\}$	**Laplace Transform** $X(s) = L\{x(t)\}$	**Condition**
e^{-at}	$\frac{1}{s+a}$	
$\sin\omega_n t$	$\frac{a}{s^2+\omega_n^2}$	
$\cos\omega_n t$	$\frac{s}{s^2+\omega_n^2}$	
$\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{1}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)$	Underdamped Motion ($\zeta < 1$)
$\frac{\omega_n^2}{s}\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$	$1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \arccos(\zeta))$	Underdamped Motion ($\zeta < 1$)
$e^{-at}x(t)$	$X(s+a)$	
$\delta(t-a)$	e^{-as}	
$u(t-a)x(t)$	$e^{-as}X(s)$	

Table 4.1: Often used Laplace transforms.

5. Multiple-Degree-of-Freedom Systems

5.1 Governing Equations of a Two-Degree-of-Freedom System

In previous chapters we have only looked at systems with one changing variable x. In reality situations can hardly ever be expressed by just one variable. To investigate multiple-degree-of-freedom systems, we will first look at two-degree-of-freedom systems. An example of such a system is shown in figure 5.6.

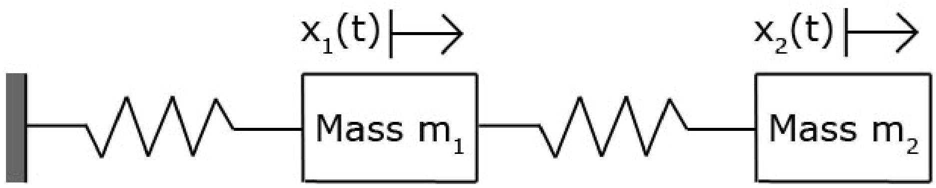

Figure 5.6: An example of a two-degree-of-freedom system.

When drawing the equations of motion for each mass, the general equations of motion can be derived. These are

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1), \tag{5.1.1}$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1). \tag{5.1.2}$$

(We are not considering damping for multiple-degree-of-freedom systems.) When solving this system, four boundary conditions are necessary. These are x_{1_0}, $\dot{x}_{1_0}$, x_{2_0} and $\dot{x}_{2_0}$.

However, writing things like this is a bit annoying. It's better to use vectors and matrices. First let's define the position vector $\mathbf{x}$, the velocity vector $\dot{\mathbf{x}}$ and the acceleration vector $\ddot{\mathbf{x}}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \qquad \text{and} \qquad \ddot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}. \tag{5.1.3}$$

We can also define the **mass matrix** (also called the **inertia matrix**) for two-degree-of-freedom cases as

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}. \tag{5.1.4}$$

Finally we also need the **stiffness matrix**. For our example system, this matrix is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}. \tag{5.1.5}$$

Now we can write the system of differential equations as

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}. \tag{5.1.6}$$

Note that both M and K are symmetric matrices (meaning that $M^T = M$ and $K^T = K$). M is symmetric because all non-diagonal terms are simply zero. K is symmetric due to Newton's third law.

We now want to find the equation of motion $\mathbf{x}(t)$ for the system of differential equations. To get it, we need to solve equation 5.1.6. There are multiple ways to do this. We'll discuss two ways.

5.2 First Method to find the Equation of Motion

The first method we will be discussing is usually the simplest method for hand calculation. It is therefore quite suitable for applying on examinations. Computers, however, don't prefer this method.

Let's suppose our solution has the form $\mathbf{x}(t) = \mathbf{u}e^{i\omega t}$. Filling this in into the differential equation will give

$$\left(K - \omega^2 M\right) \mathbf{u}e^{i\omega t} = \mathbf{0}. \tag{5.2.1}$$

The exponential can't be zero. Also, if $\mathbf{u} = \mathbf{0}$, we won't have any motion either. So we need to have ω such that the matrix $\left(K - \omega^2 M\right)$ is **singular** (not invertible). In other words, its determinant must be zero. The **characteristic equation** then is

$$\det\left(K - \omega^2 M\right) = 0. \tag{5.2.2}$$

For our two-degree-of-freedom example system, this results in

$$m_1 m_2 \omega^4 - \left(m_1 k_2 + m_2 k_1 + m_2 k_2\right) \omega^2 + k_1 k_2 = 0. \tag{5.2.3}$$

From this equation four values of ω will be found, being $\pm\omega_1$ and $\pm\omega_2$. These are the **natural frequencies** of the system. So although a one-degree-of-freedom has only one natural frequency, a two-degree-of-freedom system has 2 natural frequencies. Multiple-degree-of-freedom systems have even more natural frequencies.

The corresponding (nonzero) vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ can now be found using

$$\left(K - M\omega_1^2\right) \mathbf{u_1} = \mathbf{0} \qquad \text{and} \qquad \left(K - M\omega_2^2\right) \mathbf{u_2} = \mathbf{0}. \tag{5.2.4}$$

Only the direction of the vectors $\mathbf{u}$ can be derived from the above relations. Their magnitudes may be chosen arbitrarily, although they are often **normalized** such that $||\mathbf{u}|| = 1$. The final equation of motion is then given by

$$\mathbf{x}(t) = A_1 \sin\left(\omega_1 t + \phi_1\right) \mathbf{u}_1 + A_2 \sin\left(\omega_2 t + \phi_2\right) \mathbf{u_2}. \tag{5.2.5}$$

The values of A_1, ϕ_1, A_2 and ϕ_2 now need to be determined from the initial conditions.

5.3 Second Method to find the Equation of Motion

There is another way to find the equation of motion. Before we discuss this method, we first have to make some definitions. We define the **matrix square root** $M^{1/2}$ of M such that

$$M^{1/2} M^{1/2} = M \qquad \Rightarrow \qquad M^{1/2} = \begin{bmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{bmatrix}. \tag{5.3.1}$$

This matrix also has an inverse $\left(M^{1/2}\right)^{-1} = M^{-1/2}$. Let's define the vector $\mathbf{q}$ such that

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t). \tag{5.3.2}$$

Let's assume $\mathbf{q} = \mathbf{v}e^{i\omega t}$, with $\mathbf{v}$ a constant vector. We can now rewrite equation 5.1.6 to

$$M^{-1/2} K M^{-1/2} \mathbf{v} = \widetilde{K}\mathbf{v} = \omega^2 \mathbf{v}, \tag{5.3.3}$$

where $\widetilde{K} = M^{-1/2} K M^{-1/2}$ is the **mass normalized stiffness**. If we replace ω^2 by λ in the above equation we have exactly the eigenvalue problem from linear algebra. The solutions for λ are then the **eigenvalue**s of the matrix $\widetilde{K}$ and the corresponding vectors $\mathbf{v}$ are the **eigenvectors**.

Since K is symmetric, also $\widetilde{K}$ is symmetric. All the eigenvalues are therefore real numbers and also the eigenvectors are real. Once the eigenvalues λ_1 and λ_2 are known, the natural frequencies ω_1 and ω_2 can easily be found using

$$\omega_1 = \sqrt{\lambda_1} \qquad \text{and} \qquad \omega_2 = \sqrt{\lambda_2}. \tag{5.3.4}$$

To find the corresponding vectors $\mathbf{u}$, you can use

$$\mathbf{u_1} = M^{-1/2}\mathbf{v_1} \qquad \text{and} \qquad \mathbf{u_2} = M^{-1/2}\mathbf{v_2} \tag{5.3.5}$$

The equation of motion is then once more given by

$$\mathbf{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\,\mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\,\mathbf{u_2}. \tag{5.3.6}$$

5.4 Modal Analysis

We can also find the equation of motion using **modal analysis**. In the previous paragraph we have found the eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$ of the matrix $\widetilde{K}$. These vectors are orthogonal (unless they correspond to the same eigenvalue, in which case they should be made orthogonal). If they have also been normalized (given length 1), then they form an **orthonormal set**. Now let's define the **matrix of eigenvectors** P to consist of these orthonormal eigenvectors. In an equation this is

$$P = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix}. \tag{5.4.1}$$

This matrix is an **orthogonal matrix** (as its columns are orthonormal). Such matrices have the convenient property that $P^T P = I$. Also let's define the **matrix of mode shapes** S as

$$S = M^{-1/2}P. \tag{5.4.2}$$

Furthermore we define the vector $\mathbf{r}(t)$ such that

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = M^{-1/2}P\mathbf{r}(t) = S\mathbf{r}(t). \tag{5.4.3}$$

Using all these definitions, we can rewrite the system of differential equations to

$$\ddot{\mathbf{r}}(t) + \Lambda\mathbf{r}(t) = \mathbf{0}, \tag{5.4.4}$$

where the matrix Λ is given by

$$\Lambda = P^T \widetilde{K} P = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}. \tag{5.4.5}$$

So we remain with the differential equations

$$\ddot{r}_1 + \omega_1^2 r = 0, \tag{5.4.6}$$
$$\ddot{r}_2 + \omega_2^2 r = 0. \tag{5.4.7}$$

The differential equations have been decoupled! They don't depend on each other, and therefore can be solved using simple methods. The two decoupled equations above are called the **modal equations**. Also the coordinate system $\mathbf{r}(t)$ is called the **modal coordinate system**.

To solve the modal equations, we need the initial conditions in the modal coordinate system. Usually we only know the initial conditions $\mathbf{x_0}$ and $\dot{\mathbf{x}}_\mathbf{0}$ in the normal coordinate system. We can transform these to the modal coordinate system using

$$\mathbf{r_0} = S^{-1}\mathbf{x_0} \qquad \text{and} \qquad \dot{\mathbf{r}}_\mathbf{0} = S^{-1}\dot{\mathbf{x}}_\mathbf{0}, \qquad \text{where} \qquad S^{-1} = P^T M^{1/2}. \tag{5.4.8}$$

Now we can solve for $r_1(t)$ and $r_2(t)$ and thus for $\mathbf{r}(t)$. Once we have found $\mathbf{r}(t)$ we can find the equation of motion $\mathbf{x}(t)$ using

$$\mathbf{x}(t) = S\mathbf{r}(t). \tag{5.4.9}$$