GATE PATHSHALA

Fluid Mechanics and Aerodynamics (Assignment-01: Solutions)

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Total Acceleration, Convective Acceleration, and Local Acceleration

Concept Recap

The **total acceleration** of a fluid particle in a **3D unsteady flow** is given by the **material derivative** of velocity:

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} \tag{1}$$

where:

- $\frac{\partial \mathbf{V}}{\partial t}$ is the **local acceleration** (due to unsteady effects).
- $(\mathbf{V} \cdot \nabla)\mathbf{V}$ is the **convective acceleration** (due to spatial changes in velocity).

In Cartesian coordinates, if $\mathbf{V} = (u, v, w)$, then the components of acceleration are:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \tag{2}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}, \tag{3}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}. \tag{4}$$

Acceleration Formula in Cylindrical Coordinates

The total acceleration in cylindrical coordinates is given by:

$$a_r = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}, \tag{5}$$

$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}, \tag{6}$$

$$a_z = \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}. \tag{7}$$

Question 1: Understanding Total Acceleration

A velocity field in two-dimensional flow is given as:

$$\mathbf{V} = (2xy)\mathbf{i} + (x^2 + y^2)\mathbf{j}$$

Find the total acceleration at the point $(1, 2)$.

Solution:

Total acceleration is given by:

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}$$

Since the velocity field does not explicitly depend on time, the **local acceleration** is zero:

$$\frac{\partial \mathbf{V}}{\partial t} = 0$$

Now, computing the **convective acceleration**:

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right)\mathbf{V}$$

Given $u = 2xy$, $v = x^2 + y^2$, we compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 2y, \qquad \frac{\partial u}{\partial y} = 2x,$$
$$\frac{\partial v}{\partial x} = 2x, \qquad \frac{\partial v}{\partial y} = 2y.$$

Convective acceleration components:

$$\begin{aligned}
a_x &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\
&= (2xy)(2y) + (x^2 + y^2)(2x) \\
&= 4xy^2 + 2x(x^2 + y^2) \\
&= 4xy^2 + 2x^3 + 2xy^2 \\
&= 6xy^2 + 2x^3
\end{aligned}$$

$$\begin{aligned}
a_y &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \\
&= (2xy)(2x) + (x^2 + y^2)(2y) \\
&= 4x^2y + 2y(x^2 + y^2) \\
&= 4x^2y + 2yx^2 + 2y^3 \\
&= 6x^2y + 2y^3
\end{aligned}$$

Evaluating at $(1, 2)$:

$$\begin{aligned} a_x &= 6(1)(2)^2 + 2(1)^3 \\ &= 6(4) + 2(1) \\ &= 24 + 2 = 26, \end{aligned}$$

$$\begin{aligned} a_y &= 6(1)^2(2) + 2(2)^3 \\ &= 12 + 16 \\ &= 28. \end{aligned}$$

Thus, the total acceleration is:

$$\mathbf{a} = (26\mathbf{i} + 28\mathbf{j})\ \text{m/s}^2$$

Question 2: Local and Convective Acceleration Components in cylindrical coordinates in 2D

A fluid velocity in cylindrical coordinates is given as:

$$V_r = 3r^2 t, \quad V_\theta = 2r\theta$$

Find the **local acceleration** and **convective acceleration** at $(r, \theta) = (2, \frac{\pi}{4})$ when $t = 1$.

Solution:

Local Acceleration

The local acceleration components are:

$$a_r^{(local)} = \frac{\partial V_r}{\partial t}, \qquad a_\theta^{(local)} = \frac{\partial V_\theta}{\partial t}.$$

Computing the partial derivatives:

$$\frac{\partial V_r}{\partial t} = 3r^2, \qquad \frac{\partial V_\theta}{\partial t} = 0.$$

At $r = 2, \theta = \frac{\pi}{4}, t = 1$:

$$\begin{aligned} a_r^{(local)} &= 3(2)^2 = 12, \\ a_\theta^{(local)} &= 0. \end{aligned}$$

Convective Acceleration

The convective acceleration components in cylindrical coordinates are:

$$a_r^{(conv)} = V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r},$$
$$a_\theta^{(conv)} = V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r}.$$

Compute Partial Derivatives

$$\frac{\partial V_r}{\partial r} = 6rt,$$
$$\frac{\partial V_r}{\partial \theta} = 0,$$
$$\frac{\partial V_\theta}{\partial r} = 2\theta,$$
$$\frac{\partial V_\theta}{\partial \theta} = 2r.$$

Compute $a_r^{(conv)}$

$$\begin{aligned} a_r^{(conv)} &= (3r^2t)(6rt) + \frac{(2r\theta)}{r}(0) - \frac{(2r\theta)^2}{r} \\ &= 18r^3t^2 - \frac{4r^2\theta^2}{r}. \end{aligned}$$

Evaluating at $r = 2, \theta = \frac{\pi}{4}, t = 1$:

$$\begin{aligned} a_r^{(conv)} &= 18(2)^3(1)^2 - \frac{4(2)^2(\pi/4)^2}{2} \\ &= 144 - \frac{16\pi^2}{32} \\ &= 144 - \frac{\pi^2}{2}. \end{aligned}$$

Approximating $\pi^2 \approx 9.8696$:

$$a_r^{(conv)} \approx 144 - 4.9348 = 139.07.$$

Compute $a_\theta^{(conv)}$

$$\begin{aligned} a_\theta^{(conv)} &= (3r^2t)(2\theta) + \frac{(2r\theta)}{r}(2r) + \frac{(3r^2t)(2r\theta)}{r} \\ &= 6r^2t\theta + 4r\theta + 6r^2t\theta \\ &= 12r^2t\theta + 4r\theta. \end{aligned}$$

Evaluating at $r = 2, \theta = \frac{\pi}{4}, t = 1$:

$$\begin{aligned} a_\theta^{(conv)} &= 12(2)^2\frac{\pi}{4} + 4(2)\frac{\pi}{4} \\ &= 12(4)\frac{\pi}{4} + 8\frac{\pi}{4} \\ &= 12\pi + 2\pi = 14\pi. \end{aligned}$$

Approximating $14\pi \approx 43.98$.

Total Acceleration

Summing the local and convective accelerations:

$$\begin{aligned} a_r &= a_r^{(local)} + a_r^{(conv)} = 12 + 139.07 = 151.07, \\ a_\theta &= a_\theta^{(local)} + a_\theta^{(conv)} = 0 + 43.98 = 43.98. \end{aligned}$$

Thus, the corrected total acceleration is:

$$\mathbf{a} \approx (151.07\mathbf{e_r} + 43.98\mathbf{e}_\theta) \text{ m/s}^2.$$

Question 3

Find the Total Acceleration in a Given Flow Field **Problem:** Given the velocity field:

$$\mathbf{V} = (u, v, w) = (x^2 t, y t^2, z t^3),$$

find the total acceleration $\mathbf{a}$ at the point $(x, y, z, t) = (1, 2, 3, 1)$.

Solution

Compute Local Acceleration $\frac{\partial \mathbf{V}}{\partial t}$:

$$\frac{\partial u}{\partial t} = x^2, \quad \frac{\partial v}{\partial t} = 2yt, \quad \frac{\partial w}{\partial t} = 3zt^2.$$

At $(1, 2, 3, 1)$:

$$\frac{\partial u}{\partial t} = 1, \quad \frac{\partial v}{\partial t} = 4, \quad \frac{\partial w}{\partial t} = 9.$$

Compute Convective Acceleration $(\mathbf{V} \cdot \nabla)\mathbf{V}$:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = (x^2 t)(2xt) + (yt^2)(0) + (zt^3)(0).$$

At $(1, 2, 3, 1)$:

$$(1 \cdot 1)(2 \cdot 1) + (2 \cdot 1)(0) + (3 \cdot 1)(0) = 2.$$

Similarly,

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = (x^2 t)(0) + (yt^2)(t^2) + (zt^3)(0).$$

At $(1, 2, 3, 1)$:

$$(1 \cdot 1)(0) + (2 \cdot 1)(1) + (3 \cdot 1)(0) = 2.$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = (x^2t)(0) + (yt^2)(0) + (zt^3)(t^3).$$

At $(1, 2, 3, 1)$:

$$(1 \cdot 1)(0) + (2 \cdot 1)(0) + (3 \cdot 1)(1) = 3.$$

Total Acceleration:

$$a_x = 1 + 2 = 3, \quad a_y = 4 + 2 = 6, \quad a_z = 9 + 3 = 12.$$

Thus,

$$\mathbf{a} = (3, 6, 12).$$

Question 4

A fluid has a **radial velocity field** in cylindrical coordinates:

$$V_r = Are^{-t}, \quad V_\theta = 0, \quad V_z = Bzt.$$

Find the **total acceleration** in cylindrical coordinates.

Solution

Compute Local Acceleration

The local acceleration terms are:

$$\frac{\partial V_r}{\partial t} = -Are^{-t}, \quad \frac{\partial V_\theta}{\partial t} = 0, \quad \frac{\partial V_z}{\partial t} = Bz.$$

Compute Convective Acceleration

Radial Component:

$$V_r\frac{\partial V_r}{\partial r} = (Are^{-t})\frac{\partial}{\partial r}(Are^{-t}) = (Are^{-t})(Ae^{-t}) = A^2re^{-2t}.$$

Since $V_\theta = 0$, the term $-\frac{V_\theta^2}{r}$ is zero, and the convective acceleration from V_z is also zero.

Thus, the total acceleration in :

Radial direction:

$$a_r = -Are^{-t} + A^2re^{-2t}.$$

Tangential Component:

$$a_\theta = 0.$$

Axial Component:

$$V_r\frac{\partial V_z}{\partial r} = (Are^{-t})\frac{\partial}{\partial r}(Bzt) = (Are^{-t})(0) = 0,$$

$$\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} = 0,$$

$$V_z\frac{\partial V_z}{\partial z} = (Bzt)\frac{\partial}{\partial z}(Bzt) = (Bzt)(Bt) = B^2zt^2.$$

Summing all contributions:

$$a_z = Bz + B^2zt^2.$$

Final Answer

$$\mathbf{a} = \begin{bmatrix} -Are^{-t} + A^2re^{-2t} \\ 0 \\ Bz + B^2zt^2 \end{bmatrix}.$$

Question 5: Total Acceleration

The velocity field in two dimensions is given as:

$$\mathbf{V} = (u, v) = (x^2t, yt^2)$$

We need to find the total acceleration:

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}$$

Compute the Local Acceleration

The local acceleration is given by:

$$\frac{\partial \mathbf{V}}{\partial t}$$

For $u = x^2t$:

$$\frac{\partial u}{\partial t} = x^2$$

For $v = yt^2$:

$$\frac{\partial v}{\partial t} = 2yt$$

Thus,

$$\frac{\partial \mathbf{V}}{\partial t} = (x^2, 2yt)$$

Compute the Convective Acceleration

The convective acceleration is given by:

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\mathbf{V}$$

For the x-component (u):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$= (x^2 t)\frac{\partial}{\partial x}(x^2 t) + (yt^2)\frac{\partial}{\partial y}(x^2 t)$$

$$= (x^2 t)(2xt) + (yt^2)(0)$$

$$= 2x^3 t^2$$

For the y-component (v):

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

$$= (x^2 t)\frac{\partial}{\partial x}(yt^2) + (yt^2)\frac{\partial}{\partial y}(yt^2)$$

$$= (x^2 t)(0) + (yt^2)(t^2)$$

$$= yt^4$$

Thus,

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = (2x^3 t^2, yt^4)$$

Compute the Total Acceleration

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}$$

$$= (x^2, 2yt) + (2x^3 t^2, yt^4)$$

$$= (x^2 + 2x^3 t^2, 2yt + yt^4)$$

Final Answer

$$\mathbf{a} = \left(x^2 + 2x^3 t^2, 2yt + yt^4\right)$$

Question 6: Total Acceleration in Polar Coordinates

A velocity field in polar coordinates is given as:

$$V_r = r^2 t, \quad V_\theta = rt^2$$

Compute the total acceleration components.

Total Acceleration Formula in 2D

The total acceleration in polar coordinates is given by:

Radial Component:

$$a_r = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r}$$

Tangential Component:

$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r}$$

Computing Partial Derivatives

Time Derivatives:

$$\frac{\partial V_r}{\partial t} = r^2, \quad \frac{\partial V_\theta}{\partial t} = 2rt$$

Spatial Derivatives:

$$\frac{\partial V_r}{\partial r} = 2rt, \quad \frac{\partial V_\theta}{\partial r} = t^2$$
$$\frac{\partial V_r}{\partial \theta} = 0, \quad \frac{\partial V_\theta}{\partial \theta} = 0$$

Computing Acceleration Components

Radial Acceleration:

$$\begin{aligned} a_r &= \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} \\ &= r^2 + (r^2 t)(2rt) + 0 - \frac{(rt^2)^2}{r} \\ &= r^2 + 2r^3t^2 - rt^4 \end{aligned}$$

Tangential Acceleration:

$$\begin{aligned} a_\theta &= \frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} \\ &= (2rt) + (r^2 t)(t^2) + 0 + \frac{(r^2 t)(rt^2)}{r} \\ &= 2rt + r^2t^3 + r^3t^3/r \\ &= 2rt + r^2t^3 + r^2t^3 \\ &= 2rt + 2r^2t^3 \end{aligned}$$

Final Answer

$$a_r = r^2 + 2r^3t^2 - rt^4$$
$$a_\theta = 2rt + 2r^2t^3$$

This gives the total acceleration vector in polar coordinates.